Indian Statistical Institute, Bangalore Centre
M.Math I Year, Second Semester
Solution set of Mid-Sem Examination 2013-2014
Functional Analysis

1. Show that the vector space of all polynomials on [0,1], is not a Banach space under any norm.

   Proof. Since a Banach space can't have a denumerable Hamel basis and $p_n(t) = t^n$ for $t \in [0,1]$, $n \in \mathbb{N} \cup \{0\}$ forms a Hamel basis for the vector space $\mathcal{P}[0,1]$ of all polynomials on $[0,1]$, $\mathcal{P}[0,1]$ is not a Banach space under any norm. □

2. Let H be a complex Hilbert space. Let $x_1, \ldots, x_k$ be a set of orthonormal vectors. For any $x \in H$, and complex numbers $z_1, \ldots, z_k$, show that

$$\|x - \sum_{j=1}^{k} z_j x_j\| \geq \|x - \sum_{j=1}^{k} \langle x, x_j \rangle x_j\|$$

   Proof. By the fact that every orthonormal set in a Hilbert space can be extended to an orthonormal basis, we can extend the orthonormal set $\{x_1, \ldots, x_k\}$ to orthonormal basis $\{u_\alpha\}$ (it may happen that $\{u_\alpha\}$ is an uncountable set). But we know that the set $E_x = \{u_\alpha : \langle x, u_\alpha \rangle \neq 0\}$ is a countable set. Say $E_x = \{u_1, u_2, \ldots\}$ and by using the Fourier expansion for $x \in H$, we can write

$$x = \sum_{n=1}^{\infty} \langle x, u_n \rangle u_n.$$

   Observe that

$$\begin{aligned} x - \sum_{j=1}^{k} z_j x_j &= \sum_{n=1}^{\infty} \langle x, u_n \rangle u_n - \sum_{j=1}^{k} z_j x_j \\ &= \sum_{u_n \neq x_j} \langle x, u_n \rangle u_n + \sum_{j=1}^{k} \langle x, x_j \rangle x_j - \sum_{j=1}^{k} z_j x_j \\ &= \sum_{u_n \neq x_j} \langle x, u_n \rangle u_n + \sum_{j=1}^{k} [\langle x, x_j \rangle - z_j] x_j. \qquad (1) \end{aligned}$$

Here we would like to note that a scalar $\langle x, x_k\rangle$ may be equal to zero, but then it will not effect the summand. Further,

$$x - \sum_{j=1}^{k}\langle x, x_j\rangle x_j = \sum_{u_n \neq x_j} \langle x, u_n\rangle u_n. \tag{2}$$

From equations (1) and (2), we have

$$\|x - \sum_{j=1}^{k} z_j x_j\|^2 = \sum_{u_n \neq x_j} |\langle x, u_n\rangle|^2 + \sum_{j=1}^{k} |\langle x, x_j\rangle - z_j|^2, \tag{3}$$

and

$$\|x - \sum_{j=1}^{k}\langle x, x_j\rangle x_j\|^2 = \sum_{u_n \neq x_j} |\langle x, u_n\rangle|^2. \tag{4}$$

Thus equations (3) and (4) yield the following:

$$\|x - \sum_{j=1}^{k} z_j x_j\| \geq \|x - \sum_{j=1}^{k}\langle x, x_j\rangle x_j\|. \qquad \square$$

3. Let X be a normed linear space and let $T : X \to X$ be a linear map such that $x_n \to 0$ implies, $T(x_n)_{n\geq 1}$ is bounded. Show that T is continuous.

   Proof. Suppose if possible T is not a continuous map. Then T is not bounded on $U(0; \frac{1}{n}) = \{x \in X : \|x\| < \frac{1}{n}\}$ for each $n \in \mathbb{N}$. Then for each $n \in \mathbb{N}$ there exists $x_n \in U(0; \frac{1}{n})$ such that $\|Tx_n\| > n$. Since $\|x_n\| < \frac{1}{n}$ for each $n \in \mathbb{N}$, $x_n \to 0$, and we have already noted that $T(x_n)_{n\geq 1}$ is not bounded as $\|Tx_n\| > n$ for each $n \in \mathbb{N}$. Thus we arrive at a contradiction. Hence T is continuous. $\square$

4. Let $\{X_n\}_{n\geq 1}$ be a sequence of Banach spaces. Let

$$Y = \{\{x_n\}_{n\geq 1} : x_n \in X_n \text{ and } \lim_{n\to\infty} \|x_n\|_{X_n} = 0\}.$$

   Show that Y is a Banach space with the norm

$$\|\{x_n\}_{n\geq 1}\|_Y = \sup\{\|x_n\|_{X_n} : n \in \mathbb{N}\}.$$

Proof. We will check only the condition that every Cauchy sequence in Y is a convergent sequence, remaining conditions of a Banach space can be checked very easily. Assume that $\{x^m\}_{m\geq 1}$ is a Cauchy sequence in Y, where for each $m \in \mathbb{N}$, $x^m = \{x_n^m\}_{n\geq 1}$. Then by the definition of the Cauchy sequence given $\epsilon > 0$ there exists $\ell \in \mathbb{N}$ such that

$$\begin{aligned} & \|x^m - x^k\|_Y < \epsilon \text{ for all } m, k \geq \ell \\ \Rightarrow\ & \sup_{n\geq 1} \|x_n^m - x_n^k\|_{X_n} < \epsilon \text{ for all } m, k \geq \ell \\ \Rightarrow\ & \|x_n^m - x_n^k\|_{X_n} < \epsilon \text{ for all } m, k \geq \ell, n \in \mathbb{N}. \end{aligned} \tag{5}$$

Thus $\{x_n^m\}_{m\geq 1}$ is a Cauchy sequence in the Banach space X_n for each $n \in \mathbb{N}$. So there exists $x_n \in X_n$ for each $n \in \mathbb{N}$ such that $\|x_n^m - x_n\|_{X_n} \to 0$ as $m \to \infty$ for each $n \geq 1$. Now fix m and taking $k \to \infty$ in equation (5), we have

$$\sup_{n\geq 1} \|x_n^m - x_n\|_{X_n} < \epsilon \text{ for all } m \geq \ell. \tag{6}$$

Our next claim is to show that $\{x_n\}_{n\geq 1} \in Y$. For that purpose we need to prove that $\lim_{n\to\infty} \|x_n\|_{X_n} = 0$. Since $x^\ell \in Y$, $\lim_{n\to\infty} \|x_n^\ell\|_{X_n} = 0$. Then there exists $n_0 \in \mathbb{N}$ such that $\|x_n^\ell\|_{X_n} < \epsilon$ for all $n \geq n_0$. Observe that

$$\begin{aligned} & \|x_n\|_{X_n} - \|x_n^\ell\|_{X_n} \leq \|x_n^\ell - x_n\|_{X_n} < \epsilon \text{ (from equation (6))} \\ \Rightarrow\ & \|x_n\|_{X_n} - \|x_n^\ell\|_{X_n} < \epsilon \\ \Rightarrow\ & \|x_n\|_{X_n} < \|x_n^\ell\|_{X_n} + \epsilon \\ \Rightarrow\ & \|x_n\|_{X_n} < \epsilon + \epsilon \text{ for all } n \geq n_0. \end{aligned}$$

Thus $\lim_{n\to\infty} \|x_n\|_{X_n} = 0$. Hence from equation (6) the Cauchy sequence $\{x^m\}_{m\geq 1}$ in Y converges to the element $\{x_n\}_{n\geq 1} \in Y$. $\square$

5. Let Y be as in question 4. Let $x_n^* \in X_n^*$ be a sequence such that $\sum_{n=1}^{\infty} \|x_n^*\| < \infty$. Define $T : Y \to \mathbb{R}$ by $T(\{x_n\}_{n\geq 1}) = \sum_{n=1}^{\infty} x_n^*(x_n)$. Show that T is well-defined, linear map. Show that $\|T\| = \sum_{n=1}^{\infty} \|x_n^*\|$.

Proof. First we prove T is well-defined, for that we need to show the series $\sum_{n=1}^{\infty} x_n^*(x_n)$ converges in $\mathbb{R}$. This follows from the following argument:

$$\begin{aligned} |\sum_{n=1}^{\infty} x_n^*(x_n)| &\leq \sum_{n=1}^{\infty} |x_n^*(x_n)| \\ &\leq \sum_{n=1}^{\infty} \|x_n^*\| \|(x_n)\|_{X_n} \\ &\leq \sup\{\|x_n\|_{X_n} : n \in \mathbb{N}\} \sum_{n=1}^{\infty} \|x_n^*\| \\ &= \|\{x_n\}_{n\geq 1}\|_Y \sum_{n=1}^{\infty} \|x_n^*\| < \infty. \end{aligned} \tag{7}$$

Since x_n^* is a linear map for each $n \in \mathbb{N}$, T is also a linear map. In equation (7), we have already shown that T is bounded and

$$\|T\| \leq \sum_{n=1}^{\infty} \|x_n^*\|.$$

Next we prove the above inequality in other way. We know that $\|x_n^*\| = \sup_{\|y\|_{X_n} \leq 1} |x_n^*(y)|$, so given $\epsilon > 0$, there exists $y_n \in X_n$ with $\|y_n\|_{X_n} \leq 1$ for each $n \in \mathbb{N}$ such that

$$\|x_n^*\| - \frac{\epsilon}{2^n} < |x_n^*(y_n)|.$$

Define $z_n = \dfrac{x_n^*(y_n)}{|x_n^*(y_n)|} y_n$ if $|x_n^*(y_n)| \neq 0$, otherwise $z_n = 0$. Then $\|z_n\|_{X_n} \leq 1$ for each $n \in \mathbb{N}$. Further define $x^m = \{z_1, z_2, \ldots, z_m, 0, 0, \ldots\}$ for each $m \in \mathbb{N}$. Clearly $x^m \in Y$ and $\|x^m\|_Y \leq 1$ for each $m \in \mathbb{N}$. Now for each $m \in \mathbb{N}$

$$T(x^m) = \sum_{n=1}^{m} x_n^*(z_n) = \sum_{n=1}^{m} |x_n^*(y_n)| > \sum_{n=1}^{m} \|x_n^*\| - \sum_{n=1}^{m} \frac{\epsilon}{2^n}.$$

Since $\|x^m\|_Y \leq 1$, $T(x^m) = |T(x^m)| \leq \|T\|$. Then above equation yields

$$\sum_{n=1}^{m} \|x_n^*\| - \sum_{n=1}^{m} \frac{\epsilon}{2^n} < \|T\|$$

for each $m \in \mathbb{N}$. Taking $m \to \infty$, we have

$$\sum_{n=1}^{\infty} \|x_n^*\| - \epsilon \leq \|T\|.$$

Since $\epsilon > 0$ is arbitrary, we have $\sum_{n=1}^{\infty} \|x_n^*\| \leq \|T\|$. Thus $\|T\| = \sum_{n=1}^{\infty} \|x_n^*\|$. $\square$

6. Consider the Hilbert space, $\ell^2 = \{\{\alpha_n\}_{n\geq 1} : \sum_{n=1}^{\infty} |\alpha_n|^2 < \infty\}$. Let $f : \ell^2 \to \mathbb{C}$ be a linear map, that is not continous. Show that $\ker f$ is a dense subspace of ℓ^2.

Proof. Since $f : \ell^2 \to \mathbb{C}$ is not continuous, f is not bounded for $U(0; \frac{1}{n}) = \{x \in \ell^2 : \|x\|_\ell^2 < \frac{1}{n}\}$ for each $n \in \mathbb{N}$. So there exists $x_n \in U(0; \frac{1}{n})$ such that $|f(x_n)| > n$ for each $n \in \mathbb{N}$. Note that $x_n \to 0$ and $|f(x_n)| \to \infty$ as $n \to \infty$. Let $x \in \ell^2$. Define $y_n = x - \dfrac{f(x)}{f(x_n)} x_n$ for each $n \in \mathbb{N}$. Then using the linearity of f we can see that $y_n \in \ker f$ for each $n \in \mathbb{N}$. Since $x_n \to 0$ and $|f(x_n)| \to \infty$ as $n \to \infty$, $y_n \to x$ as $n \to \infty$. Therefore $x \in \overline{\ker f}$ and which implies $\overline{\ker f} = \ell^2$. Hence $\ker f$ is a dense subspace of ℓ^2. $\square$

7. Let $\{f_n\}_{n\geq 1} \subset L^1[0,1]$ be such that $\int_0^1 |f_n| d\lambda \to 0$. Show that $\exists$ a subsequence $\{f_{n_k}\}_{k\geq 1}$ of $\{f_n\}_{n\geq 1}$ such that $f_{n_k} \to 0$ a.e..

Proof. We denote $[0,1]$ by X. Let $\epsilon > 0$. For each $n \in \mathbb{N}$ let $E_n = \{x \in X : |f_n(x)| \geq \epsilon\}$, and note that each E_n is a measurable set of finite measure. Now, since $\epsilon\chi_{E_n} \leq |f_n|$ holds for each n, it follows that, $\epsilon\mu(E_n) \leq \int_0^1 |f_n| d\mu$ also holds for each n. Thus, $\lim_{n\to\infty} \mu(E_n) = 0$ (i.e., f_n convergences in measure to the zero function and denoted by $f_n \xrightarrow{\mu} 0$). Since $f_n \xrightarrow{\mu} 0$, we can find $n_1 < n_2 < \ldots$ such that

$$\mu\{x \in X : |f_n(x)| \geq \frac{1}{k}\} < \frac{1}{2^k},\ \forall n \geq n_k.$$

Define $F_k = \{x \in X : |f_{n_k}(x)| \geq \frac{1}{k}\}$ and $H_m = \bigcup_{k=m}^{\infty} F_k$. Then we have

$$\mu(F_k) < \frac{1}{2^k} \quad \text{and} \quad \mu(H_m) \leq \sum_{k=m}^{\infty} \mu(F_k) < \sum_{k=m}^{\infty} \frac{1}{2^k} = \frac{1}{2^{m-1}}.$$

Further define

$$Y = \bigcap_{m=1}^{\infty} H_m.$$

Then $\mu(Y) \leq \mu(H_m) < \dfrac{1}{2^{m-1}}$ for every $m \geq 1$, so we have $\mu(Y) = 0$. If $x \notin Y$, then $x \notin H_m$ for some m. Hence $x \notin F_k$ for all $k \geq m$, which implies

$$|f_{n_k}(x)| < \frac{1}{k},\ \forall k \geq m.$$

Thus $f_{n_k}(x) \to 0$ for all $x \notin Y$. Since Y has measure zero, we therefore have pointwise convergence of f_{n_k} to 0 almost everywhere. $\square$

8. Show that the space of continuous functions with compact support on $\mathbb{R}$ is not a Banach space w.r.t. the supremum norm.

Proof. For each $n \in \mathbb{N}$ define the function

$$g_n(x) = \begin{cases} 0 & \text{if } -\infty < x \leq -n, \\ x+n & \text{if } -n \leq x \leq -n+1, \\ 1 & \text{if } -n+1 \leq x \leq n-1, \\ -x+n & \text{if } n-1 \leq x \leq n, \\ 0 & \text{if } n \leq x < \infty. \end{cases}$$

Then $g_n \in C_c(\mathbb{R})$ for each $n \in \mathbb{N}$. Take $f(x) = e^{-x^2}, x \in \mathbb{R}$. Note that $fg_n \in C_c(\mathbb{R})$ for all $n \in \mathbb{N}$ and $\|fg_n - f\|_\infty \to 0$ but $f \notin C_c(\mathbb{R})$. So $C_c(\mathbb{R})$ is not a Banach space w.r.t. the supremum norm. $\square$